Simple Formulas for Stability of Strict Equilibria in Best Experienced Payoff Dynamics*

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Dedicated to the memory of Bill Sandholm

Abstract

We consider a family of population game dynamics known as Best Experienced Payoff Dynamics, under which each revising agent tests some of her possible strategies a fixed number of times, with each play of each strategy being against a newly drawn opponent, and chooses the strategy whose total payoff was highest, breaking ties according to a given tie-breaking rule. Strict Nash equilibria are rest points of these dynamics, but need not be stable. We provide some simple formulas and algorithms to determine the stability or instability of strict Nash equilibria. *JEL* classification numbers: C72, C73.

Keywords: Best Experienced Payoff; Procedural rationality; Stability

1. Introduction

Most dynamics in Evolutionary Game Theory can be neatly seen as a combination of a *population game* and a *revision protocol* (Sandholm, 2010). The *population game* assigns to each population state a vector of payoffs, *one* for each strategy in the population. The *revision protocol* specifies how agents, using the payoff assigned to each strategy, update their current strategy. A crucial assumption embedded in this framework is that, at any

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population state, there is *one single* payoff assigned to each strategy. In population games where agents are matched to play a symmetric normal form game, the payoff assigned to each strategy is often the expected payoff the agent will obtain when using that strategy. But how can agents know this expected payoff? Unless there is complete matching, agents somehow know the exact population state, or agents are explicitly communicated the precise expected payoff for each strategy, it seems unrealistic to assume that they will all share exactly the same expectations for any given strategy. From this point of view, it is noteworthy that many evolutionary dynamics from the economics literature are informationally demanding in one important respect: they require agents to be fully informed about the population's current aggregate behavior. This assumption seems rather strong in the large-populations contexts to which evolutionary models are most naturally applied.

In many situations, it seems more natural to assume that agents acquire information by interacting with only a *sample* of the population, rather than assuming that they have access to accurate statistics of the whole population. There are two distinct lines of research that follow this approach while keeping the assumption that agents respond optimally to the information they have.

The first line assumes that agents take samples of the actions being played in the population, and they use these samples to make inferences about the distribution of actions in the whole population, and to best respond to the estimates thus formed. This is the approach followed by Sandholm (2001), Kosfeld et al. (2002), Osborne and Rubinstein (2003), Kreindler and Young (2013), Oyama et al. (2015), Heller and Mohlin (2018), Salant and Cherry (2020), and Sawa and Wu (2021). Under this approach, note that agents must be aware of the population game they are playing, so they can best reply to their point estimates of the population distribution of actions.

A second approach –significantly less demanding on agents' informational and computational skills– was pioneered by Osborne and Rubinstein (1998) and Sethi (2000). Here, revising agents try out a subset of the available strategies by playing them against randomly drawn counterparts, and then choose the strategy that performed best in the test. Crucially, each game is played against new randomly drawn counterparts, so sub-optimal strategies may be selected in the test if they happened to be lucky in the random sampling of co-players. In this approach, note that agents do not even need to know that they are playing a game. Agents who follow this revision protocol are called *procedurally rational agents* (Osborne and Rubinstein, 1998), and the evolutionary dynamics they produce are the so-called *Best Experienced Payoff* (BEP) dynamics (Sandholm et al., 2019). These dynamics are the main object of study in this paper.

Procedurally rational agents and their associated equilibria have been used in a variety of applications, including consumer choice procedures and product pricing strategies (Spiegler, 2006a), markets with asymmetric information (Spiegler, 2006b), trust and delegation of control (Rowthorn and Sethi, 2008), the Traveler's Dilemma (Berkemer, 2008), market entry (Chmura and Güth, 2011), ultimatum bargaining (Miękisz and Ramsza, 2013), use of common-pool resources (Cárdenas et al., 2015), contributions to public goods (Mantilla et al., 2020), the Centipede game (Sandholm et al., 2019; Izquierdo and Izquierdo, 2021), the Prisoner's Dilemma (Arigapudi et al., 2021), and coordination problems (Izquierdo et al., 2022). Sethi (2021) studies the equilibria of these processes in symmetric, finitely repeated games, with several applications.

In BEP dynamics, strict Nash equilibria of a game correspond to states that are rest points, but they may not be stable. Sandholm et al. (2020), building on Sethi's (2000) pioneering work, provide several sufficient conditions for instability and for asymptotic stability of strict equilibria under BEP dynamics. Arigapudi et al. (2021) refine one of the most general sufficient stability conditions in Sandholm et al. (2020), providing a tighter one. While many of the stability and instability conditions in Sandholm et al. (2020) are really simple and can be immediately checked from the payoffs of the game, the most general stability condition (Theorem 2 II in Arigapudi et al. (2021)), and the most general instability condition (Proposition 5.4 in Sandholm et al. (2020)) are –if taken at face value– actually difficult to check, as they state a condition over all sets in a certain power set, or require finding a subset that satisfies some condition. Here we show that these general stability and instability conditions can be checked by conducting a simple analysis, whose complexity is equivalent to carrying out an iterated elimination of dominated strategies, and which admits a simple interpretation. We also provide some tighter tests for specific BEP dynamics.

The rest of the paper is structured as follows. Section 2 contains a short introduction to Best Experienced Payoff processes and their dynamics. In Section 3 we summarize previous results on stability of strict equilibria, indicating also the new contributions in this paper. Section 4 presents the new stability tests and formulas, and in Section 5 we state some conclusions. The proofs, and some additional information, have been grouped in an appendix.

2. Best experienced payoff protocols and dynamics

Although all our results can be easily extended to asymmetric games played in p populations, for notational simplicity we keep our presentation to p-player symmetric games played in one population. Following Sandholm et al. (2020), we consider a unit-mass population of agents who are matched to play a symmetric p-player normal form game $G = \{S, U\}$. This game is defined by a strategy set $S = \{1, \ldots, n\}$, and a payoff function $U\colon S^p \to \mathbb{R}$, where $U(i; j_1, \ldots, j_{p-1})$ represents the payoff obtained by a strategy i player whose opponents play strategies $j_1, \ldots, j_{p-1}$. Our symmetry assumption requires that the value of U not depend on the ordering of the last $p-1$ arguments. When $p = 2$, we sometimes write U_{ij} instead of $U(i; j)$.

Aggregate behavior in the population is described by a *population state* x in the simplex $X = \{x \in \mathbb{R}^S_+ \colon \sum_{i \in S} x_i = 1\}$, with x_i representing the fraction of agents in the population using strategy $i \in S$. The standard basis vector $e_i \in X$ represents the pure (monomorphic) state at which all agents play strategy i.

We consider Best Experienced Payoff (BEP) protocols defined by a triple (τ, κ, β). Under BEP protocols, agents occasionally revise their current strategy by conducting tests of alternative strategies.

The first parameter, namely the *test-set rule* τ, indicates how the set of strategies to be tested is chosen. Specifically, here we consider the test-set rule τ^α, under which the revising agent, when considering whether to change his current strategy, will also test other $\alpha - 1$ randomly selected strategies in S (besides testing his current strategy). Naturally, $\alpha \in \mathbb{N}$ and $1 < \alpha \leq n$. If all the strategies in S are tested, i.e. if $\alpha = n$, we have the test-all rule, denoted by τ^{all}.

The second parameter, called the *number of trials* $\kappa \in \mathbb{N}$, specifies the number of times that each strategy will be played in the test. Thus, each strategy in the test set will be played by the revising agent over κ matches, with each match requiring a new independent sampling of $p - 1$ co-players.

The last parameter in the BEP protocol, namely the *tie-breaking rule* β, indicates the rule used to decide which strategy is selected when the best result (the best total payoff) in the tests is obtained by more than one strategy. We will omit the last parameter when our results are independent of the tie-breaking rule. Otherwise, we will focus on two tie-breaking rules. The uniform-if-tie rule, β^{unif}, selects any of the strategies that obtain the best total payoff in the tests, each of these strategies with equal probability. This is the rule that has been considered in almost all cases in the literature. The stick-if-tie rule, β^{stick}, chooses to keep using the current strategy if it obtains the best total payoff in the tests, and, otherwise, it breaks ties by random uniform selection among the strategies that obtained the best total payoff.

Well-known results of Benaïm and Weibull (2003) show that the behavior of a large but finite population following the procedure above is closely approximated by the solution of the associated *mean dynamic*, a differential equation which describes the expected motion of the population from each state. This mean dynamic for BEP processes is (Sethi, 2000):

$$\dot{x}_i = w_i(x) - x_i$$

where $w_i(x)$ is the probability with which strategy i is selected by a revising agent, i.e., the probability that it is tested, it obtains the best total payoff, and, if there are ties, it is selected by the tie-breaking rule. The calculation of the term $w_i(x)$, i.e. the mean dynamic, for BEP($\tau^{\alpha}, \kappa, \beta$) processes, was formalized by Sandholm et al. (2020).

3. Stability and instability under BEP dynamics. Antecedents and contribution

Consider a strict strategy s in a symmetric p-player game, i.e., a strategy s such that the strategy profile $(s, s, ..., s)$ is a strict Nash equilibrium of the game. Following Osborne and Rubinstein's (1998) pioneering study of rest points of the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic, and Sethi's (2000) stability analysis of the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic, Sandholm et al. (2020) show that the linear stability analysis of an equilibrium state e_s – a monomorphic state where all players use the same strict strategy s – under any BEP(τ, κ, β) dynamic, can be reduced to the analysis of an $n \times n$ matrix $V^{\kappa,s} = (v_{ij}^{\kappa,s})$ of total payoffs $v_{ij}^{\kappa,s}$, defined by

$$v_{ij}^{\kappa,s} = (\kappa - 1)U(i; s, s, ..., s) + U(i; j, s, ..., s)$$

To simplify the notation, we will drop the superindex s when it is clear that we are refering to a specific equilibrium strategy s, in which case we will use V^{κ} and v_{ij}^{κ}. The Jacobian of the dynamics at the equilibrium s can be calculated from the terms in V^{κ}. The term v_{ij}^{κ} is the total payoff to strategy i when, over its κ trials, it meets exclusively players using the strict Nash strategy s, except in one trial, where exactly one of the $(p-1)$ co-players uses strategy j. The reason why these are the only relevant payoffs for a linear stability analysis is that, in the proximity of the strict equilibrium, where $x_s = 1 - \epsilon$, the probability of any random sample of $\alpha\,\kappa\,(p-1)$ co-players with more than one co-player choosing a strategy other than s is $O(\epsilon^2)$.

Thus, when α strategies are tested, the relevant sampling events are:

i) Those in which all the $\alpha\,\kappa\,(p-1)$ randomly sampled co-players use strategy s. In this case, a test of strategy s provides the total payoff v_{ss}^{κ} and a test of strategy $i \neq s$ provides the total payoff v_{is}^{κ}. Since s is a strict Nash strategy, $v_{ss}^{\kappa} > v_{is}^{\kappa}$, so, if strategy s

is in the test set, then it will be selected.

ii) Those in which all but one of the sampled co-players use strategy s and exactly one co-player (the "deviating co-player") uses strategy $j \neq s$. Assuming all strategies are tested:

- If the deviating co-player is met when testing strategy s, the total payoffs are v^κ_{sj} and $\{v^\kappa_{is}\}_{i \in S \setminus \{s\}}$. Defining $S_2 \equiv \text{argmax}_{i \neq s} v^\kappa_{is} = \text{argmax}_{i \neq s} U(i; s, s, ..., s)$, we have that either the selected strategy belongs to S_2, or the selected strategy is s, depending on the comparison of v^κ_{sj} and $v^\kappa_{ts} \equiv \max_{i \neq s} v^\kappa_{is}$. In case of equality, the tie-breaking rule would apply.
- If the deviating co-player is met when testing strategy $i \neq s$, the total payoffs are v^κ_{ss}, v^κ_{ij} and $\{v^\kappa_{ks}\}_{k \in S \setminus \{s,i\}}$. Since every element in $\{v^\kappa_{ks}\}_{k \in S \setminus \{s,i\}}$ is less than v^κ_{ss}, the selected strategy is either s or i, depending on the comparison of v^κ_{ss} and v^κ_{ij}. In case of equality, the tie-breaking rule would apply.

To analyze the stability of a strict equilibrium state e_s, Sandholm et al. (2020) consider a change of variables that takes e_s to the origin $\mathbf{0}$ (by eliminating the coordinate x_s, given that $\sum_{i=1}^n x_i = 1$) and show that the Jacobian of the dynamics at the origin is $DW(\mathbf{0}) = DW^+(\mathbf{0}) - I_{(n-1)\times(n-1)}$, where $DW^+(\mathbf{0})$ is a matrix of non-negative terms that can be easily calculated from the terms in V^κ, following the previous discussion.

3.1 Instability

A series of instability results (i.e. sufficient instability conditions) can be derived from the analysis of V^κ by considering that the Perron-Frobenius eigenvalue of $DW^+(\mathbf{0})$ is at least as large as the Perron-Frobenius eigenvalue of any principal submatrix of $DW^+(\mathbf{0})$, which is in turn bounded from below by the minimum sum of the elements in each of its columns (or rows). If the Perron-Frobenius eigenvalue of $DW^+(\mathbf{0})$ is greater than 1, then $DW(\mathbf{0})$ has a real positive eigenvalue[1] and, consequently, e_s is unstable. A general condition that guarantees instability following this approach is provided by Proposition 5.4 (ii) in Sandholm et al. (2020), which states that e_s is linearly unstable under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic if, for some nonempty $J \subseteq S \setminus \{s\}$,

$$(p-1)\kappa \frac{\alpha-1}{n-1}\left(\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + \mathbf{1}[S_2 \subseteq J]\,\mathbf{1}[v^\kappa_{sj} < v^\kappa_{ts}]\right) > 1 \text{ for all } j \in J,$$

[1]If λ is an eigenvalue of $DW^+(\mathbf{0})$, then $\lambda - 1$ is an eigenvalue of $DW(\mathbf{0})$.

where $\mathbf{1}[\cdot]$ denotes the indicator function. As, under BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics (i.e., for $\alpha = n$) and given a subset of strategies $J \subseteq S \setminus \{s\}$, this result considers a tight bound on the column sums of the submatrix of $DW^{+}(\mathbf{0})$ corresponding to the strategies in J,[2] this is, up to our knowledge, the most general available result that guarantees instability under BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics (for any tie-breaking rule) with either $\kappa > 1$ or $p > 2$.

3.2 Stability

A series of stability results (i.e. sufficient stability conditions) can also be derived from the analysis of V^{κ} by considering that, if $DW^{+}(\mathbf{0})$ is a triangular matrix, its eigenvalues are its diagonal elements. If the eigenvalues of $DW^{+}(\mathbf{0})$ are all less than one, then the eigenvalues of $DW(\mathbf{0})$ are all negative and, consequently, e_s is stable. This can be used to show, for instance, that, under any BEP($\tau^{\alpha}, \kappa, \beta$) dynamics, any strict equilibrium state is asymptotically stable if the number of trials is larger than a certain threshold (Sandholm et al., 2020, Corollary 5.8).

Under BEP($\tau^{\text{all}}, \kappa$) dynamics, the most general condition that guarantees that the Jacobian of $DW^{+}(\mathbf{0})$ can be arranged as a triangular matrix whose diagonal elements are 0 is the existence of an ordering of the strategies in S such that, for all $i, j \neq s$ with $i \geq j$ we have: $v^{\kappa}_{ss} > v^{\kappa}_{ij}$ and, if $i \in S_2$, $v^{\kappa}_{sj} > v^{\kappa}_{is}$. This is a refinement of Proposition 5.9 in Sandholm et al. (2020) that can be shown to be equivalent to the sufficient condition for asymptotic stability in Theorem 2 (II) in Arigapudi et al. (2021).

Arigapudi et al. (2021) focus on the BEP($\tau^{\text{all}}, \kappa$) dynamic and on a family of games that satisfy a specific genericity requirement, which here we term κ-*generic* games (Arigapudi et al., 2021, Definition 4). They show that their sufficient stability condition for asymptotic stability of of strict Nash equilibria is both sufficient and necessary in κ-generic games with either more than two players ($p > 2$) or more than one test of each strategy ($\kappa > 1$). However, this stability condition is difficult to check if followed literally, since it involves testing a requirement on each and every set in the power set of $S \setminus \{s\}$. The requirement of having a κ-*generic* game can also be too stringent in practical cases, as it may not be satisfied even by two-player games with generic payoff matrices. As an illustration, none of the more than 20 numeric examples in Osborne and Rubinstein (1998), Sethi (2000), Sandholm et al. (2019, 2020), Sethi (2021) and Arigapudi et al. (2021) are κ-generic.

3.3 Contribution

In this paper we:

[2]See note at the beginning of appendix A.2.

i) Show that the general sufficient condition for instability of strict equilibria indicated above (Sandholm et al., 2020, Proposition 5.4 (ii)), which applies under any BEP(τ^{α}, κ) dynamics, can be checked using a simple algorithm. The complexity of this algorithm is equivalent to performing an iterated elimination of dominated strategies.

ii) Show that a similarly simple algorithm can be used to check the general sufficient condition for asymptotic stability of strict equilibria under BEP($\tau^{\text{all}}, \kappa$) dynamics indicated in Section 3.2[3], i.e., the most general condition that guarantees, under any tie-breaking rule, a triangular Jacobian $DW(\mathbf{0})$ with diagonal values (i.e. eigenvalues) equal to -1. We also provide a tighter stability test under the specific tie-breaking rule β^{stick}, a rule that favors stability under BEP($\tau^{\text{all}}, \kappa$) dynamics.

iii) Discuss conditions under which the sufficient condition for asymptotic stability in ii) is also necessary for stability, for different BEP($\tau^{\text{all}}, \kappa$) dynamics. This extends the results by Arigapudi et al. (2021) by removing the constraint that the game be *κ-generic*.

4. Stability and instability tests

4.1 s-stabilizing and potentially s-stabilizing strategies

In this section we define s-stabilizing and potentially s-stabilizing strategies in subsets $J \subseteq S \setminus \{s\}$. Informally, a s-stabilizing strategy in J is a strategy that, under a BEP($\tau^{\text{all}}, \kappa$) dynamic, does not contribute to the growth of the fraction of players using the strategies in J, when the population state is close to the strict equilibrium state s. In contrast, if a strategy is not potentially s-stabilizing, it is associated to at least some minimum contribution to the growth of the fraction of players using the strategies in J, when the population state is close to the strict equilibrium state s, under any BEP(τ^{α}, κ) dynamic.

Definition (s-stabilizing and potentially s-stabilizing strategies)**.** Let S_2 be the set of strategies that obtain the second-best payoff, v^{κ}_{ts}, when playing against s-players, i.e., $S_2 \equiv \text{argmax}_{i \neq s}\, v^{\kappa}_{is} = \text{argmax}_{i \neq s}\, U(i; s, s, ..., s)$, and $v^{\kappa}_{ts} \equiv \max_{i \neq s} v^{\kappa}_{is}$. Let J be a non-empty set $J \subseteq S \setminus \{s\}$. A strategy $j \in J$ is s-stabilizing in J, for a number of trials κ, if

- $v^{\kappa}_{ij} < v^{\kappa}_{ss}$ for all $i \in J$, and
- If $S_2 \cap J \neq \emptyset$, then $v^{\kappa}_{sj} > v^{\kappa}_{ts}$.

[3]As indicated before, this is equivalent to the sufficient condition for asymptotic stability in Arigapudi et al. (2021), Theorem 2, II.

A strategy $j \in J$ is potentially s-stabilizing in J, for a number of trials κ, if

- $v^{\kappa}_{ij} \leq v^{\kappa}_{ss}$, for all $i \in J$, and
- If $S_2 \subseteq J$, then $v^{\kappa}_{sj} \geq v^{\kappa}_{ts}$. □

Clearly, every s-stabilizing strategy in J is potentially s-stabilizing in J. To understand the previous conditions, consider a test of each strategy by a revising agent who, when sampling the required $n\,\kappa\,(p-1)$ co-players, meets just once a deviating co-player not using strategy s, but using strategy $j \in J$ instead. The condition $v^{\kappa}_{ij} < v^{\kappa}_{ss}$ guarantees that, if the deviating j-player is met when testing strategy $i \in J$, the total payoff v^{κ}_{ij} to strategy i is less than the total payoff v^{κ}_{ss} to strategy s, so strategy s is selected. Similarly, the condition $((S_2 \cap J \neq \emptyset) \Rightarrow v^{\kappa}_{sj} > v^{\kappa}_{ts})$ guarantees that, if the deviating j-player is met when testing strategy s (in which case the two highest total payoffs are v^{κ}_{sj} and v^{κ}_{ts}), no strategy $i \in J$ is selected. Under any BEP($\tau^{\text{all}}, \kappa$) dynamic, if a strategy j is s-stabilizing, then its total (positive or destabilizing) contribution to the submatrix –corresponding to the strategies in J– of the Jacobian of the dynamics at the equilibrium (specifically, its contribution to the corresponding submatrix of $DW^{+}(\mathbf{0})$), in the column corresponding to j, is zero. This fact can be used to provide sufficient conditions for the asymptotic stability of the equilibrium. Note that if a strategy is s-stabilizing in J for a number of trials κ_0, then it is s-stabilizing in J for any $\kappa > \kappa_0$.

If a strategy j is not potentially s-stabilizing, then its (positive or destabilizing) contribution to the submatrix of the Jacobian of the dynamics corresponding to the strategies in J, in the column corresponding to j, is guaranteed to be above a certain threshold value, under any BEP(τ^{α}, κ) dynamic. If no strategy in J is potentially s-stabilizing, the fact that the sum of the positive contributions in every column of a principal submatrix of the Jacobian is above a threshold value can be used to obtain a lower bound for the Perron–Frobenius eigenvalue of the matrix, and to guarantee instability of the equilibrium. Note that if a strategy is not potentially s-stabilizing in J for a number of trials κ_0, then it is not potentially s-stabilizing in J for any $\kappa < \kappa_0$.

4.2 Instability under BEP(τ^{α}, κ) dynamics

Our first proposition shows that a tight sufficient test for instability of strict equilibria under any BEP(τ^{α}, κ) dynamics can be carried out by analyzing the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$. Although the process of iterated elimination may be considered evident, a formal description can be found in appendix A.1. All the proofs have been relegated to appendix A.2.

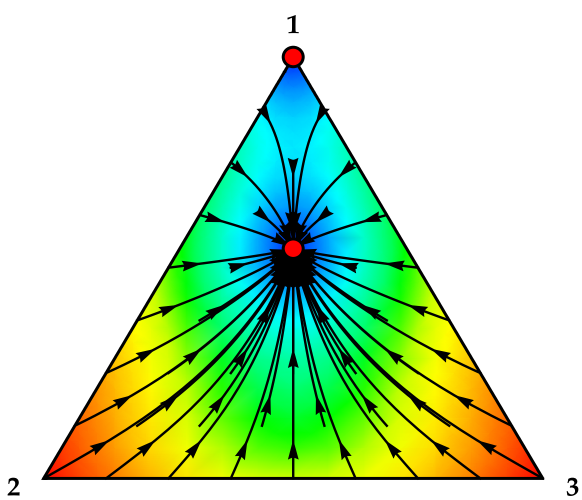
Proposition 4.1. *Let s be a strict equilibrium. Consider a BEP(τ^{α}, κ) dynamic with $\kappa = \kappa_0 > \frac{n-1}{(p-1)(\alpha-1)}$. If some strategy survives the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is unstable for any κ satisfying $\frac{n-1}{(p-1)(\alpha-1)} < \kappa \leq \kappa_0$.*

Corollary 4.2. *Let s be a strict equilibrium. Consider a BEP($\tau^{\mathrm{all}}, \kappa$) dynamic with $\kappa = \kappa_0$. If some strategy survives the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is unstable for any κ with $1 < \kappa \leq \kappa_0$, and, if $p > 2$, also for $\kappa = 1$.*

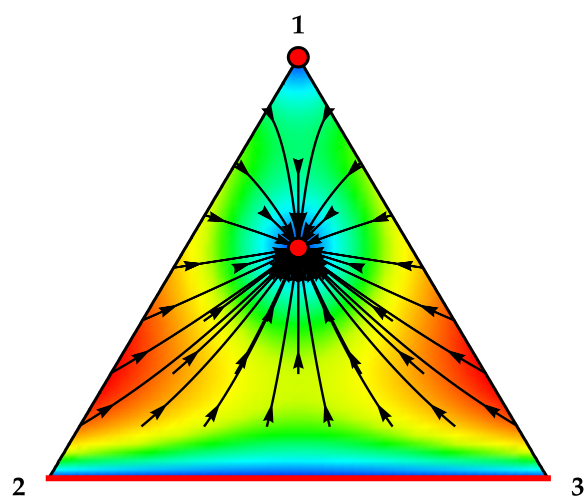
Example 4.1. Consider the game with payoff matrix

$$A = V^{\kappa=1} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \text{ which leads to } V^{\kappa=2} = \begin{pmatrix} 6 & 3 & 3 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix}. \tag{1}$$

Corollary 4.2 shows that the equilibrium state e_1 is unstable under BEP($\tau^{\mathrm{all}}, \kappa = 2$) dynamics. This can be proved by noting that, for $\kappa = 2$, strategies 2 and 3 survive the iterated elimination of potentially 1-stabilizing strategies, since none of them is potentially 1-stabilizing in $J = S \setminus \{s\} = \{2, 3\}$. This is so because, for $s = 1$ and $j \in J$, we have that $S_2 = \{2, 3\} \subseteq J$ but $v_{1j}^{\kappa=2} = 3 < 4 = v_{t1}^{\kappa=2}$. However, for $\kappa = 2$, this game satisfies the necessary conditions for asymptotic stability in Theorem 2 in Arigapudi et al. (2021), which are not sufficient in this case, since the game is not κ-generic. Thus, Corollary 4.2 (and Proposition 4.1, more generally) can be used to prove the instability of strict equilibria on which Theorem 2 in Arigapudi et al. (2021) remains silent.



(i) tie-breaking rule β^{unif}



(ii) tie-breaking rule β^{stick}

Figure 1: BEP($\tau^{\mathrm{all}}, 2, \beta$) dynamics in the game of Example 4.1 for two tie-breaking rules: β^{unif} (left) and β^{stick} (right). All figures in this paper can be easily replicated using *EvoDyn-3s* (Izquierdo et al., 2018).

Figure 1 shows the BEP($\tau^{\text{all}}, 2, \beta$) dynamics in the game of Example 4.1 for two tie-breaking rules: β^{unif} (left) and β^{stick} (right). As proved above for any tie-breaking rule, it can be seen that state e_1 is unstable under both dynamics. ◆

4.3 Asymptotic stability under BEP($\tau^{\text{all}}, \kappa$) dynamics

Proposition 4.3. *Let s be a strict equilibrium. Consider any BEP($\tau^{\text{all}}, \kappa$) dynamic with $\kappa = \kappa_0$. If no strategy survives the iterated elimination of s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is asymptotically stable for any $\kappa \geq \kappa_0$.*

For a fixed κ, the stability condition in Proposition 4.3 can be shown to be equivalent to the stability condition in Arigapudi et al. (2021) [Theorem 2, II],[4] so the former can be seen as a quick and easy way of checking the latter. In terms of the complexity of checking these conditions according to their formulation, Proposition 4.3 involves checking the existence of s-stabilizing strategies in at most $n-1$ subsets of S, while a direct check of the stability condition in Arigapudi et al. (2021) involves checking an existence condition in 2^{n-1} subsets of S (in all the subsets of $S \setminus \{s\}$). If, for instance, the number of strategies is $n = 11$, the difference would be checking 10 subsets using Proposition 4.3 versus checking $2^{10} = 1024$ subsets otherwise.

Example 4.2. Consider the coordination game with payoff matrix

$$\begin{pmatrix} U_{11} & 0 & 0 & \dots & 0 \\ 0 & U_{22} & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & U_{(n-1)(n-1)} & 0 \\ 0 & 0 & \dots & 0 & U_{nn} \end{pmatrix}, \tag{2}$$

with $U_{ss} > 0$ for all $s \in S$, so all strategies are strict Nash strategies. In this game, for $i, j \in S \setminus \{s\}$ and $i \neq j$, we have $v_{ss}^{\kappa,s} = \kappa U_{ss}$, $v_{sj}^{\kappa,s} = (\kappa-1)U_{ss}$, $v_{ii}^{\kappa,s} = U_{ii}$ and $v_{ij}^{\kappa,s} = 0$. Therefore, strategy $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if the following two conditions are satisfied:

- $v_{ij}^{\kappa} < v_{ss}^{\kappa}$ for all $i \in J$ $\quad \Leftrightarrow U_{jj} < \kappa U_{ss} \quad \Leftrightarrow \kappa > \dfrac{U_{jj}}{U_{ss}}$.

[4] It is not difficult to show that the sufficient stability condition in Arigapudi et al. (2021) can be equivalently formulated in terms of iterated elimination of strategies that are not weakly supported (according to their definition) by any other strategy. This is so because if a strategy j is not weakly supported by any strategy in a set J that includes j, then j is not weakly supported by any strategy in any subset of J that includes j.

- If $S_2 \cap J \neq \emptyset$ then $v_{sj}^{\kappa} > v_{ts}^{\kappa}$ $\quad\Leftrightarrow (\kappa - 1)U_{ss} > 0 \quad\Leftrightarrow \kappa > 1.$

Thus, $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if $\kappa > \max\left(\frac{U_{jj}}{U_{ss}}, 1\right)$. Similarly, it is easy to check that strategy $j \in J \subseteq S \setminus \{s\}$ is potentially s-stabilizing in J if and only if $\kappa \geq \frac{U_{jj}}{U_{ss}}$.

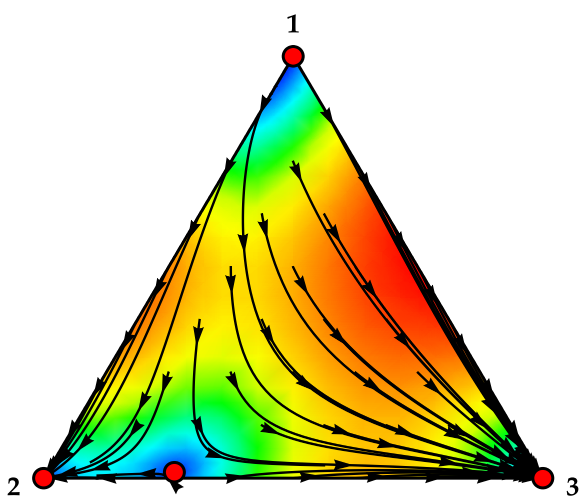
Now, let $U_{\max} = \max_{i \in S} U_{ii}$ be the highest possible payoff and let $S^{\max} = \{i \in S \mid U_{ii} = U_{\max}\}$ be the set of strategies that obtain the highest possible payoff in the game when playing against themselves.

Applying Proposition 4.3, we can deduce that, for any strict strategy $s \in S$, state e_s is asymptotically stable under any BEP($\tau^{\text{all}}, \kappa$) for every $\kappa > \frac{U_{\max}}{U_{ss}}$, since this condition guarantees that all strategies are s-stabilizing, so no strategy survives the iterated elimination of s-stabilizing strategies. In particular, if $s \in S^{\max}$, e_s is asymptotically stable for every $\kappa > \frac{U_{\max}}{U_{\max}} = 1$.

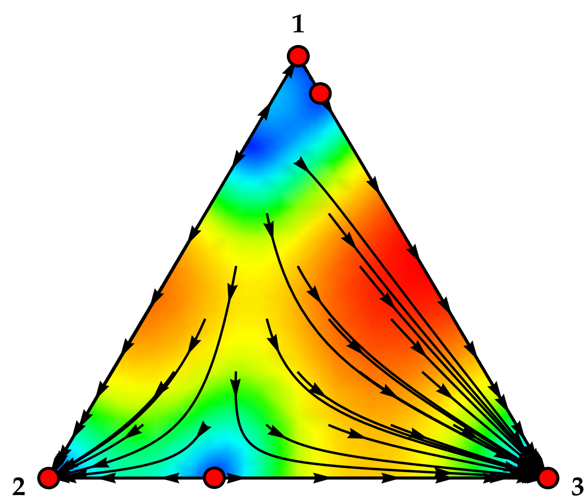
Applying Corollary 4.2, we can deduce that if $s \notin S^{\max}$, state e_s is unstable under any BEP($\tau^{\text{all}}, \kappa$) for every $1 < \kappa < \frac{U_{\max}}{U_{ss}}$, since this condition guarantees that any strategy $i \in S^{\max}$ is not potentially s-stabilizing in any subset that contains it, so it survives the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$.

So, to sum up, in coordination game (2) with $U_{ss} > 0$ for all $s \in S$, under any BEP($\tau^{\text{all}}, \kappa$) with $\kappa > 1$, e_s is asymptotically stable for $\kappa > \frac{U_{\max}}{U_{ss}}$ and e_s is unstable for $1 < \kappa < \frac{U_{\max}}{U_{ss}}$.

The stability of e_s in the remaining cases, i.e. for $\kappa = 1$ and for $\kappa = \frac{U_{\max}}{U_{ss}}$ (if $\frac{U_{\max}}{U_{ss}} \in \mathbb{N}$), depends on the tie-breaking rule.



(i) $\kappa = 2$



(ii) $\kappa = 4$

Figure 2: Coordination game (2) with $n = 3$ strategies, where $U_{ii} = i$, under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, for $\kappa = 2$ (left) and $\kappa = 4$ (right).

Figure 2 illustrates these results by showing the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in the coordination game (2) with $n = 3$ strategies and $U_{ii} = i$. For $\kappa = 2$, e_1 is unstable (since $1 < \kappa < \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$), e_2 is asymptotically stable (since $\kappa > \frac{U_{\max}}{U_{ss}} = \frac{3}{2} = 1.5$), and e_3 is asymptotically stable (since $s = 3 \in S^{\max}$ and $\kappa > 1$). For $\kappa \geq 4$, e_1 becomes asymptotically stable too (since $\kappa > \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$). ◆

4.4 Stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics

In this section we study whether the lack of fulfillment of the sufficient condition for asymptotic stability in Proposition 4.3 can guarantee instability. Arigapudi et al. (2021) show that, for BEP($\tau^{\text{all}}, \kappa$) dynamics with either $\kappa > 1$ or $p > 2$, a sufficient stability condition that is equivalent to Proposition 4.3, is both sufficient and necessary in κ-generic games. However, the requirement of being κ-generic can be quite restrictive in practice, as pointed out in Section 3.

Here we remove the genericity condition and focus on BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in any game, given that this is the BEP dynamics considered in most previous studies in the literature. In the next section, we will also consider BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics, as this alternative tie-breaking rule can be regarded as more natural in many cases.

For BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, we show that the sufficient condition for asymptotic stability in Proposition 4.3 is also necessary for stability for any $\kappa > n$; more tightly, for any $\kappa > \frac{|S_2|+1}{p-1}$. If the second-best payoff when playing against s-players is obtained by a single strategy (i.e., if $|S_2| = 1$),[5] then this property holds for any $\kappa > 2$ (for any κ, if $p > 3$).

While the sufficient condition for instability in Proposition 4.1 will usually be tighter than the lack of fulfillment of the stability condition in Proposition 4.3, our next result (i.e. Proposition 4.4) is relevant because it shows that in BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, beyond some small values of κ, and as κ grows, we will find either permanent asymptotic stability or a single transition from instability to permanent asymptotic stability. A transition from stability to instability can only happen within the small values of κ indicated in the proposition.

Proposition 4.4. *Let s be a strict equilibrium and let $S_2 = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$. Consider any BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic with $\kappa = \kappa_0$. If no strategy survives the iterated elimination of s-stabilizing strategies in $S \setminus \{s\}$, state e_s is asymptotically stable for any $\kappa \geq \kappa_0$. Otherwise, it is unstable for any κ with $\frac{|S_2|+1}{p-1} < \kappa \leq \kappa_0$, and also for any $\kappa > \frac{2}{p-1}$ satisfying $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} < \kappa \leq \kappa_0$.*

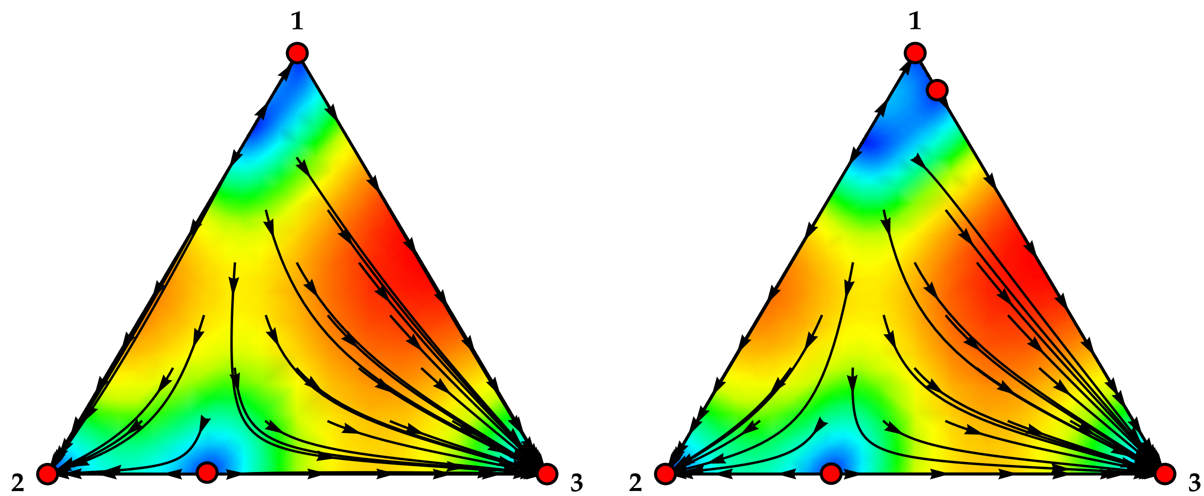
[5] Note that the condition $|S_2| = 1$ is much weaker than the condition that a game has to satisfy in order to be κ-generic.

Example 4.3. Consider the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic on the coordination game with payoff matrix (2), with $U_{ss} > 0$ for all $s \in S$. Recall that $U_{\max} = \max_{i\in S} U_{ii}$ and $S^{\max} = \{i \in S \mid U_{ii} = U_{\max}\}$.

In addition to what we inferred in Example 4.2 for any BEP($\tau^{\text{all}}, \kappa$) dynamic, applying Proposition 4.4 we can address the stability of $e_{s\notin S^{\max}}$ for $\kappa = \frac{U_{\max}}{U_{ss}}$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$). We could not do this using Corollary 4.2 because this stability depends on the tie-breaking rule. Here we deduce that if $s \notin S^{\max}$, state e_s is unstable under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) for any $\kappa \leq \frac{U_{\max}}{U_{ss}}$, assuming $\kappa > 2$.

In Example 4.2 we showed that, in this game, $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if $\kappa > \max\left(\frac{U_{jj}}{U_{ss}}, 1\right)$. Thus, if $s \notin S^{\max}$ and $\kappa \leq \frac{U_{\max}}{U_{ss}}$, any strategy $i \in S^{\max}$ survives the iterated elimination of s-stabilizing strategies in $S \setminus \{s\}$ so, applying Proposition 4.4 and noting that $\frac{v^1_{ss} - \min_{j\in S\setminus\{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} = \frac{U_{ss}-0}{U_{ss}-0} = 1$ and $p = 2$, we can state that e_s is unstable for any $\kappa \leq \frac{U_{\max}}{U_{ss}}$, assuming $\kappa > \frac{2}{p-1} = 2$.

Figure 2 shows the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in the coordination game (2) with $n = 3$ strategies and $U_{ii} = i$. For $\kappa = 3$, e_1 is unstable (since $\kappa \leq \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$), while for $\kappa \geq 4$, e_1 is asymptotically stable (since $\kappa > \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$). ◆



(i) $\kappa = 3$ (ii) $\kappa = 4$

Figure 3: Coordination game (2) with $n = 3$ strategies, where $U_{ii} = i$, under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, for $\kappa = 3$ (left) and $\kappa = 4$ (right).

4.5 Stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics

For BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics, here we provide an improved sufficient condition for asymptotic stability, tighter than Proposition 4.3, and prove that this sufficient condition for asymptotic stability is also necessary for stability for any $\kappa > \frac{|S_2|}{p-1}$. If the second-best payoff when playing against s-players is obtained by a single strategy (i.e., $|S_2| = 1$), then this condition holds for any $\kappa > 1$ (for any κ, if $p > 2$). To show this, first we need to define weakly s-stabilizing strategies.

Definition (Weakly s-stabilizing strategies)**.** We say that a strategy $j \in J$ is weakly s-stabilizing in J, for a number of trials κ, if

- $v^{\kappa}_{ij} \leq v^{\kappa}_{ss}$ for all $i \in J$, and
- If $S_2 \cap J \neq \emptyset$, then $v^{\kappa}_{sj} \geq v^{\kappa}_{ts}$. □

Any s-stabilizing strategy in J is weakly s-stabilizing in J, so if the iterated elimination of s-stabilizing strategies in $S_2 \setminus \{s\}$ eliminates all strategies (proving stability under BEP($\tau^{\text{all}}, \kappa$) dynamics), so does the iterated elimination of weakly s-stabilizing strategies. The second process, however, can prove stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics in additional cases. We illustrate this fact in Example 4.4 (also, compare Figure 3(i) vs Figure 4(ii)).

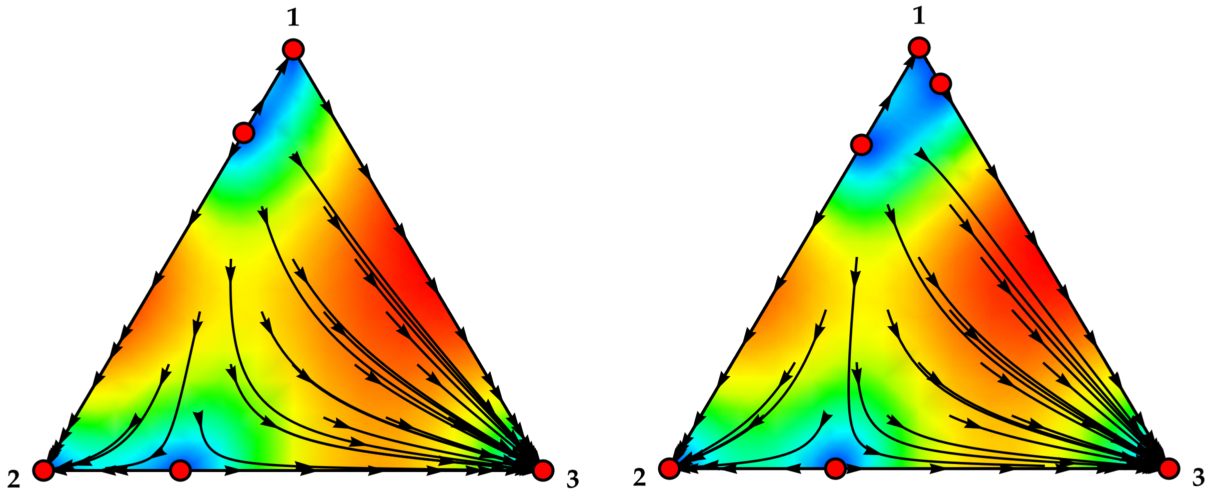
Proposition 4.5. *Let s be a strict equilibrium and let $S_2 = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$. Consider any BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic with $\kappa = \kappa_0$. If no strategy survives the iterated elimination of weakly s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is asymptotically stable for any $\kappa \geq \kappa_0$. Otherwise, it is unstable for any κ with $\frac{|S_2|}{p-1} < \kappa \leq \kappa_0$, and also for any $\kappa > \frac{1}{p-1}$ with $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} < \kappa \leq \kappa_0$.*

Note that, if $|S_2| = 1$, then the condition $\kappa > \frac{|S_2|}{p-1}$ holds for any $\kappa > 1$ (for any κ, if $p > 2$).

Example 4.4. Consider the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic on the coordination game with payoff matrix (2), with $U_{ss} > 0$ for all $s \in S$. Recall that $U_{\max} = \max_{i \in S} U_{ii}$ and $S^{\max} = \{i \in S \mid U_{ii} = U_{\max}\}$.

In Example 4.2 we showed that, in this game, $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if $\kappa > \max\left(\frac{U_{jj}}{U_{ss}}, 1\right)$. Following the same reasoning, it is easy to check that $j \in J \subseteq S \setminus \{s\}$ is weakly s-stabilizing in J if and only if $\kappa \geq \frac{U_{jj}}{U_{ss}}$.

Applying Proposition 4.5, we can then deduce that state e_s is asymptotically stable for every $\kappa \geq \frac{U_{\max}}{U_{ss}}$, since this condition guarantees that all strategies are weakly s-stabilizing, so no strategy survives the iterated elimination of weakly s-stabilizing strategies. In particular, if $s \in S^{\max}$, e_s is asymptotically stable for every $\kappa \geq \frac{U_{\max}}{U_{\max}} = 1$. If $s \notin S^{\max}$ and $\kappa < \frac{U_{\max}}{U_{ss}}$, any strategy $i \in S^{\max}$ is not weakly s-stabilizing, so it survives the iterated elimination of weakly s-stabilizing strategies in $S \setminus \{s\}$. Therefore, noting that $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} = \frac{U_{ss} - 0}{U_{ss} - 0} = 1$ and $p = 2$, we can state that e_s is unstable for any κ such that $\frac{1}{p-1} = 1 < \kappa < \frac{U_{\max}}{U_{ss}}$.



(i) $\kappa = 2$ (ii) $\kappa = 3$

Figure 4: Coordination game (2) with $n = 3$ strategies, where $U_{ii} = i$, under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics, for $\kappa = 2$ (left) and $\kappa = 3$ (right).

To sum up, under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$), e_s is asymptotically stable if $\kappa \geq \frac{U_{\max}}{U_{ss}}$, and unstable if $1 < \kappa < \frac{U_{\max}}{U_{ss}}$. In particular, if $s \in S^{\max}$, then e_s is asymptotically stable for every κ.

Figure 4 shows the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics in the coordination game (2) with $n = 3$ strategies and $U_{ii} = i$. For $\kappa = 2$, e_1 is unstable (since $1 < \kappa < \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$), e_2 is asymptotically stable (since $\kappa \geq \frac{U_{\max}}{U_{ss}} = \frac{3}{2} = 1.5$), and e_3 is asymptotically stable (since $s = 3 \in S^{\max}$). For $\kappa \geq 3$, e_1 is also asymptotically stable (since $\kappa \geq \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$). ◆

5. Conclusions

Strict Nash equilibria correspond to rest points under Best Experienced Payoff dynamics, but these rest points may be unstable. In this paper we provide a simple test, with a simple interpretation, that guarantees asymptotic stability under BEP($\tau^{\text{all}}, \kappa$) dynamics. We also provide a related simple test that guarantees instability of strict equilibria under the more general family of BEP(τ^{α}, κ) dynamics. Focusing on BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, which is the family of BEP dynamics most prevalent in the literature, we provide a stability test which, for values of κ above a small threshold value $\kappa_1 \leq n$, proves either asymptotic stability or, otherwise, instability. We also show that, as κ increases, and for $\kappa > n$, any strict equilibrium is either always asymptotically stable or there is a single transition from instability to asymptotic stability, within a bounded range of values of κ. Sandholm et al. (2020) provide bounds on the values of κ that guarantee asymptotic stability. Similar results are obtained for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic, for which we present an even tighter asymptotic stability test.

A. Appendix

A.1 Iterated elimination of strategies

Definition (Survivors of iterated elimination of strategies satisfying condition $\mathbb{C}$ in a finite set Ω)**.** Let $J^0 \equiv \Omega$ and define J^m recursively by $J^m = \{i \in J^{m-1} \mid i \text{ does not satisfy condition } \mathbb{C} \text{ in } J^{m-1}\}$. The (potentially empty) set $J^{|\Omega|}$ is the set of strategies that survive iterated elimination of strategies satisfying condition $\mathbb{C}$ in set Ω. An algorithm for this procedure is described in Algorithm 1.

Algorithm 1 Iterated elimination of strategies satisfying condition $\mathbb{C}$ in set Ω

$J \leftarrow \Omega$
while $\exists j \in J \mid j$ satisfies condition $\mathbb{C}$ in J **do**
  $J \leftarrow J \setminus \{j \in J \mid j \text{ satisfies condition } \mathbb{C} \text{ in } J\}$
end while ▷ J at the end is the set of all surviving strategies after iterated elimination

A.2 Proofs

Note. Bound on the Perron-Frobenius eigenvalue of $DW^+(\mathbf{0})$ under BEP($\tau^{\text{all}}, \kappa$) dynamics, based on a submatrix of $DW^+(\mathbf{0})$.

Under BEP($\tau^{\text{all}}, \kappa$) dynamics, the inflow (positive) terms in column j of $DW^+(\mathbf{0})$ are associated to $\mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}]$, $\mathbf{1}[v^\kappa_{ts} > v^\kappa_{sj}]$, $\mathbf{1}[v^\kappa_{ij} = v^\kappa_{ss}]$ or $\mathbf{1}[v^\kappa_{ts} = v^\kappa_{sj}]$, when the corresponding cases hold, i.e., when the indicator function takes the value 1. The inflow associated to the last two terms, $\mathbf{1}[v^\kappa_{ij} = v^\kappa_{ss}]$ and $\mathbf{1}[v^\kappa_{ts} = v^\kappa_{sj}]$, is anyway 0 under tie-breaking rules that always select the agent's current strategy if it is among the optimal tested strategies (such as β^{stick}). In this case, the less favorable for the instability of s, the inflow (positive) terms in column j of $DW^+(\mathbf{0})$ are $(p-1)\kappa\,\mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}]$, at position $DW^+_{ij}(\mathbf{0})$, plus a total inflow of $(p-1)\kappa\,\mathbf{1}[v^\kappa_{ts} > v^\kappa_{sj}]$ distributed (according to the tie-breaking rule) among the rows of $DW^+(\mathbf{0})$ corresponding to the strategies in S_2. Consequently, given a subset $J \subseteq S \setminus \{s\}$ and considering its associated submatrix $DW^+_J(\mathbf{0})$, corresponding to the strategies in J, the largest value that we can guarantee for the sum of the terms in the column of $DW^+_J(\mathbf{0})$ corresponding to strategy j is $(p-1)\kappa \sum_{i\in J} \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}]$, plus, if $S_2 \subseteq J$, $(p-1)\kappa\,\mathbf{1}[v^\kappa_{ts} > v^\kappa_{sj}]$. For $\kappa > \frac{n-1}{(p-1)(\alpha-1)}$, it can be shown, following arguments similar to the proof of fact 2 in Arigapudi et al. (2021), that proposition 5.4 (ii) in Sandholm et al. (2020), which is based on the bound discussed here (by columns), is more general than proposition 5.4 (i), which is based on a bound by rows considering only the terms $\mathbf{1}[v^{\kappa}_{ij} > v^{\kappa}_{ss}]$. □

Proof of Proposition 4.1. Considering $\kappa = \kappa_0$, if the iterated elimination of potentially s-stabilizing strategies does not eliminate all strategies in $S \setminus \{s\}$, then there is some non-empty set $J \subseteq S \setminus \{s\}$ which does not contain any potentially s-stabilizing strategies. This implies that for every $j \in J$, either $\exists i \in J$ such that $v^{\kappa}_{ij} > v^{\kappa}_{ss}$ or ($S_2 \subseteq J$ and $v^{\kappa}_{sj} < v^{\kappa}_{ts}$). With these conditions, proposition 5.4 of Sandholm et al. (2020) guarantees instability of the strict equilibrium if $\kappa > \frac{n-1}{(p-1)(\alpha-1)}$. The extension to $\kappa < \kappa_0$ comes from the fact that if a strategy is not potentially s-stabilizing in J for a number of trials κ_0, then it is not potentially s-stabilizing in J for any $\kappa < \kappa_0$. □

Proof of Proposition 4.3. Following Sandholm et al. (2020), consider a change of variables for the population state $(x_1, x_2, ..., x_n)$ that sends the equilibrium e_s to the origin $\mathbf{0}$, by eliminating the coordinate x_s while keeping the labeling of the other coordinates. In this system, consider the Jacobian of the dynamics at the equilibrium, $DW(\mathbf{0})$. Let $DW_J(\mathbf{0})$ be the square submatrix of $DW(\mathbf{0})$ whose rows and columns correspond to the strategies in J. If j is s-stabilizing in J for $\kappa = \kappa_0$, then the column of $DW_J(\mathbf{0})$ corresponding to strategy j is made up (see Sandholm et al. (2020)) by zeros in all non-diagonal positions, with a value -1 at the diagonal position. Let $(j_1, j_2, ..., j_{n-1})$ be an ordering of the $(n-1)$ strategies in $S \setminus \{s\}$ that iteratively eliminates s-stabilizing strategies. Then the column of $DW(\mathbf{0})$ corresponding to strategy j_1 is made up by zeros in all non-diagonal positions, with a value -1 at the diagonal position. Considering the cofactor expansion of the determinant of the jacobian along the column corresponding to j_1, and denoting by $DW_{-\{j_1\}}(\mathbf{0})$ the submatrix of $DW(\mathbf{0})$ obtained by eliminating the column and row corresponding to j_1, we have that $|DW(\mathbf{0})| = (-1)\,|DW_{-\{j_1\}}(\mathbf{0})|$. Now, the column of $DW_{-\{j_1\}}(\mathbf{0})$ corresponding to strategy j_2 is made up by zeros in all non-diagonal positions, with a value -1 at the diagonal position. Proceeding secuentially with the other strategies we obtain $|DW(\mathbf{0})| = (-1)\,|DW_{-\{j_1\}}(\mathbf{0})| = (-1)^2\,|DW_{-\{j_1,j_2\}}(\mathbf{0})| = ... = (-1)^{n-1}$, i.e., all the eigenvalues of the Jacobian have negative real parts, which implies asymptotic stability of the equilibrium. The result for $\kappa \geq \kappa_0$ follows from the fact that if a strategy is s-stabilizing in J for a number of trials κ_0, then it is s-stabilizing in J for any $\kappa > \kappa_0$. □

Proof of Proposition 4.4. The stability part comes from Proposition 4.3. For the instability part, first consider $\kappa = \kappa_0$. If the iterated elimination of s-stabilizing strategies does not eliminate all strategies in $S \setminus \{s\}$, then there is some non-empty set $J \subseteq S \setminus \{s\}$ which does not contain any s-stabilizing strategies. This means that for every $j \in J$, either $\exists i \in J$ such that $v^{\kappa}_{ij} \geq v^{\kappa}_{ss}$ or ($S_2 \cap J \neq \varnothing$ and $v^{\kappa}_{sj} \leq v^{\kappa}_{ts}$). Considering this and Lemma A.1 below, which is a direct adaptation of proposition 5.4 in Sandholm et al. (2020) for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic, we have that the minimum possible value of the left hand side on Equation (3) is $(p-1)\kappa\frac{1}{|S_2|+1}$, so the condition $\kappa > \frac{|S_2|+1}{p-1}$ guarantees instability. If $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} < \kappa$, then $v^\kappa_{sj} > v^\kappa_{ts}$ for all $j \neq s$ and the minimum possible value indicated before is $(p-1)\kappa\frac{1}{2}$, so the condition $\kappa > \frac{2}{p-1}$ guarantees instability. The extension to $\kappa < \kappa_0$ comes from the fact that if a strategy is not s-stabilizing in J for a number of trials κ_0, then it is not s-stabilizing in J for any $\kappa < \kappa_0$.

Lemma A.1. *Let s be a strict equilibrium, let $S_2 = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$, and let $t \in S_2$. Under any BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic, state e_s is linearly unstable if, for some nonempty $J \subseteq S \setminus \{s\}$, the following condition holds for all $j \in J$:*

$$\begin{aligned} (3) \qquad & (p-1)\kappa \left(\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + \tfrac{1}{2} \sum_{i \in J} \mathbf{1}[v^\kappa_{ij} = v^\kappa_{ss}] \right) \\ & + (p-1)\kappa\, |S_2 \cap J| \left(\tfrac{1}{|S_2|} \mathbf{1}[v^\kappa_{sj} < v^\kappa_{ts}] + \tfrac{1}{|S_2|+1} \mathbf{1}[v^\kappa_{sj} = v^\kappa_{ts}] \right) > 1 \end{aligned}$$

□

Proof of Proposition 4.5. The stability part comes from adapting the proof of Proposition 4.3 to the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic, considering that the Jacobian $DW(\mathbf{0})$ for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic has components (Sandholm et al., 2020):

$$DW_{ij}(\mathbf{0}) = \begin{cases} (p-1)\kappa\, \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] - \mathbf{1}[j = i] & \text{if } i \notin S_2, \\ (p-1)\kappa \left(\mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + \frac{1}{|S_2|} \mathbf{1}[v^\kappa_{is} > v^\kappa_{sj}] \right) - \mathbf{1}[j = i] & \text{if } i \in S_2. \end{cases}$$

For the instability part follow the steps in the proof of Proposition 4.4, noting that if a non-empty set $J \subseteq S \setminus \{s\}$ does not contain any weakly s-stabilizing strategies, then, for every $j \in J$, either $\exists i \in J$ such that $v^\kappa_{ij} > v^\kappa_{ss}$ or ($S_2 \cap J \neq \emptyset$ and $v^\kappa_{sj} < v^\kappa_{ts}$). Note also that the equivalent of Equation (3) for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic is

$$(p-1)\kappa \left(\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + |S_2 \cap J| \left(\tfrac{1}{|S_2|} \mathbf{1}[v^\kappa_{sj} < v^\kappa_{ts}] \right) \right) > 1$$

□

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